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Dynamic flexural deformations in an ideal fibre-reinforced slab

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SUMMARY

A kinematic description is derived for plane strain deformations of unrestricted amplitude in an incompressible material ideally reinforced parallel to one axis of Lagrangian coordinates. The deformation is simply related to the configuration of one reference 'fibre'. For a slab of uniform thickness, equations of motion are derived by integration over cross-sections. They relate the motion of the central fibre to the resultant tensile and shearing loads over each cross section.

The equations predict that, in many materials, flexural waves may propagate. In this paper only elastic materials are considered. The simple wave solutions are discussed, showing how wave profiles may distort until a 'crease' forms at one surface of the slab. These creases play a role somewhat similar to shocks in gas dynamics. Some novel differences are revealed by an analysis of the equations governing deformations on either side of a crease.

1. Introduction

In the past decade there has been much interest in the mechanics of materials reinforced in one or two directions by families of 'strong' fibres. Static deformations have been analysed using many different theories (see Pipkin [1] and references cited therein), whilst dynamic disturbances have been analysed largely on the basis of small deformation theory [2] or of acceleration wave theory (see Green [3] and references cited therein). For large deformations the most fruitful approach is to consider 'ideally reinforced materials' [4]). In this theory it is assumed that the fibres prevent any extension of the material along each fibre direction. Although no real material is completely inextensible in any direction, the assumption gives a good description of materials which are highly anisotropic. It reflects the observation that in a general deformation of such materials the extension along the 'strong' directions is negligible compared to other components of the deformation. Moreover, the resulting deformation may be taken as the leading term in an asymptotic expansion procedure.

For simplicity, the present paper deals only with plane strain deformations of an incompressible, ideally reinforced material which has a reference configuration in which all fibres are straight and parallel. Pipkin and Rogers [4] have previously shown that static deformations of such materials are simply described. In this paper it is shown that dynamic disturbances may similarly be described, since the configuration at any instant is completely determined by the location of two material curves.

The 'ideal' assumption leads to the frequent occurrence of layers of concentrated load (infinite stress) on portions of the boundary which are tangential to the fibres. Although this behaviour may appear unrealistic, it is now known to be a mathematical consequence of the kinematic constraint of fibre inextensibility. The description of materials having small fibre compliance involves a singular perturbation within the theory of anisotropic elasticity, and so exhibits 'transition layers' or 'boundary layers'. Within these, the fibres are subjected to large tractions which are required in order to balance the large gradients of shear which occur near boundaries of highly anisotropic materials. Linear analysis of such layers has been given by Everstine and Pipkin [5] and by Spencer [6]. In a forthcoming paper [7] the corresponding boundary layers for finite, dynamic deformations will be obtained by asymptotic procedures. In the present paper, attention is confined to the ideal material, and to plane deformations of a slab of uniform thickness.

Section 2 is used to introduce a Lagrangian description which is particularly appropriate for materials having fibres parallel to the X_1 axis of Lagrangian coordinates. It is found that at each point the tangent and the normal to the current 'fibre direction' are particularly important directions. This motivates the change to 'fibre-normal coordinates' (s,X_2), such that at each instant the curves of constant X_2 are 'fibres', whilst s = constant denotes that plane cross-section which is normal to the fibres and for which s measures length along some reference fibre (e.g. the central fibre). A complete kinematic description of the deformation and velocity fields is obtained in Section 3, using these coordinates. The momentum equations are discussed in Section 4, where it is shown that *all kinematically admissible plane strain disturbances are also mechanically admissible*. This is a generalization of the result obtained by Pipkin and Rogers [4] for static deformations. In Section 6 the momentum equations are integrated over cross-sections of constant s taking account of the concentrated loads in the surface layers as discussed in Section 5. The resulting equations involve only two independent variables (s,t) and govern motions of the reference fibre.

Kao and Pipkin, have previously [8] derived an expression for the compressive load which causes buckling of an ideally reinforced elastic column on the basis of plane strain theory. The present analysis shows that, for slabs of elastic material, there are two real finite characteristic speeds at each cross-section where the compressive load does not exceed the buckling load. Consequently, disturbances propagate as waves through regions where the buckling load is not exceeded. However, in regions where the buckling load is exceeded the speeds become complex, and small amplitude disturbances are unstable. This result confirms the Kao and Pipkin analysis of the buckling criterion.

Section 7 is concerned with simple waves, which are exact solutions of the equations derived in Section 6. These solutions demonstrate that, as a wave propagates, the curvature may continually increase, until a 'crease' develops at one of the surfaces of the beam. A 'fan' of normals centred at this travelling crease then develops. This fan formation may be compared to the formation of shocks in gas-dynamic waves. However there is an important distinction, which is discussed in Section 8. A gas-dynamic shock may be treated mathematically as a discontinuity, across which the flow variables are related by certain algebraic equations. However, the mass within a fan is proportional to the angle of the fan, so that the momentum within a fan cannot be neglected. Consequently, the equations relating conditions at either side of the fan involve rates of change of the fan angle, and so are differential equations rather than algebraic equations. It is shown that an alternative treatment for the propagation of disturbances containing fans is as a free boundary problem, in which one of the equations derived in Section 6 is replaced, within the fan, by the condition of constant curvature of the central fibre. The boundaries of the fan are then to be determined as part of the solution.

In the final section, some special configurations are analysed explicitly.

2. Dynamic plane strain

Two dimensional Lagrangian coordinates (X_1, X_2) are taken so that fibres lie along the material curves X_2 = constant, and the slab occupies the region $-1 \leqslant X_2 \leqslant 1$. The corresponding Eulerian coordinates are (x_1, x_2), so that the configuration at all times t is given by $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$. The cartesian velocity components are $v_i = \partial x_i/\partial t$, whilst $F_{ij} = \partial x_i/\partial X_j$ are the elements of the two-dimensional deformation gradient tensor $(i,j = 1,2)$.

At each point the deformation gradient may be decomposed into two axial elongations A and B followed by a simple shear γ and a rotation through an angle θ, as in Fig. 1. The corresponding deformation gradient matrix is

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A & \gamma B \\ 0 & B \end{pmatrix}. \tag{1}$$

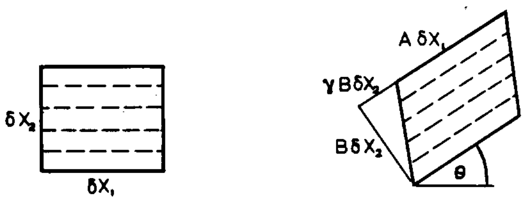

Figure 1. The parameters A, B, γ and θ in a typical state of deformation.

In an incompressible, ideally inextensible material the dilatation AB and fibre elongation A are both unity so that (1) becomes

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}. \tag{2}$$

Correspondingly the velocity $\mathbf{v}$ may be resolved into components u and v along and perpendicular to the fibres where

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{3}$$

To analyse motions $\mathbf{x} = \mathbf{x}(\mathbf{X},t)$ we look for functions γ, θ, u and v of X_1, X_2 and t for which F_{ij} and v_i satisfy the compatibility conditions

$$\frac{\partial F_{ij}}{\partial X_k} = \frac{\partial F_{ik}}{\partial X_j} \quad \text{and} \quad \frac{\partial F_{ij}}{\partial t} = \frac{\partial v_i}{\partial X_j}\,. \tag{4}$$

Substitution of expressions (2) into (4_1) gives

$$\frac{\partial(\cos\theta)}{\partial X_2} = \frac{\partial F_{11}}{\partial X_2} = \frac{\partial F_{12}}{\partial X_1} = \frac{\partial(\gamma\cos\theta - \sin\theta)}{\partial X_1}\,,$$

$$\frac{\partial(\sin\theta)}{\partial X_2} = \frac{\partial F_{21}}{\partial X_2} = \frac{\partial F_{22}}{\partial X_1} = \frac{\partial(\gamma\sin\theta + \cos\theta)}{\partial X_1}\,,$$

so that γ and θ must satisfy the equations

$$\frac{\partial\theta}{\partial X_2} - \gamma\,\frac{\partial\theta}{\partial X_1} = 0, \tag{5}$$

$$\frac{\partial(\gamma - \theta)}{\partial X_1} = 0. \tag{6}$$

3. Fibre normal coordinates

Equations (5) and (6) are kinematic constraints, which apply instantaneously at each time t. Equation (6) shows that $\gamma{-}\theta$ is uniform along each 'fibre', so that

$$\gamma = \Gamma(X_2,t) + \theta. \tag{7}$$

Similarly, equation (5) shows that the curves of constant θ are given, at each instant, by $dX_1/dX_2 = -\gamma$. From Fig. 1 it may be seen that these curves are normal to the fibres. This leads to the familiar result [1] that the fibre normals are straight, and inclined at angle $\theta + \frac{1}{2}\pi$ to the x_1-axis as shown in Fig. 2. The configuration at each time is of the type described by Pipkin and Rogers [4]. The constraints $A = 1$, $B = 1$ act instantaneously, ensuring that the sequence of configurations is a sequence of statically allowable configurations.

For simplicity, we introduce a new coordinate s which labels the fibre normals, and which measures distance $s = X_1$ along the central fibre $X_2 = 0$. Except on this fibre, the relation between s, X_1 and X_2 depends on t, so that we may write $X_1 = X(s,X_2,t)$ where

$$\frac{\partial X}{\partial X_2} = -\gamma. \tag{8}$$

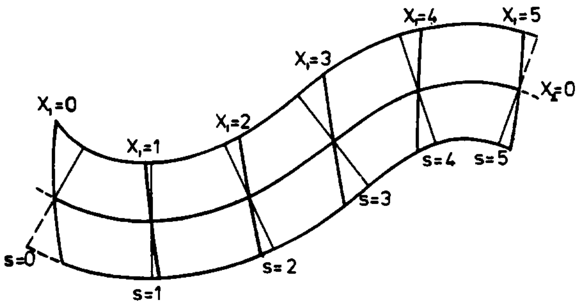

Figure 2. Deformation of a strip of material showing the relationship between the Lagrangian coordinate X_1 and the coordinate s, which is constant on fibre normals.

The quantities

$$\frac{\partial X}{\partial s} = \ell(s,X_2,t), \qquad \frac{\partial X}{\partial t} = m(s,X_2,t) \tag{9}$$

are as yet undetermined. All dependent variables are written as functions of (s,X_2,t), whilst the rules for transformation of partial derivatives are

$$\frac{\partial}{\partial X_1} \to \frac{1}{\ell}\frac{\partial}{\partial s}, \qquad \frac{\partial}{\partial X_2} \to \frac{\partial}{\partial X_2} + \frac{\gamma}{\ell}\frac{\partial}{\partial s}, \qquad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \frac{m}{\ell}\frac{\partial}{\partial s}.$$

Additionally, from (8) and (9), the compatibility conditions

$$\frac{\partial \ell}{\partial X_2} = -\frac{\partial \gamma}{\partial s}, \qquad \frac{\partial m}{\partial X_2} = -\frac{\partial \gamma}{\partial t}, \qquad \frac{\partial \ell}{\partial t} = \frac{\partial m}{\partial s} \tag{10}$$

must be imposed.

In the new coordinate system equation (5) becomes

$$\frac{\partial \theta}{\partial X_2} = 0,$$

so showing that

$$\theta = \Theta(s,t), \tag{11}$$

and consequently equation (7) becomes

$$\gamma = \Gamma(X_2,t) + \Theta(s,t). \tag{12}$$

The function $\theta = \Theta(s,t)$ is the intrinsic equation for the shape of the central fibre $X_2 = 0$ at any time t, whilst $\Gamma(X_2,t)$ measures an arbitrary extra shear which may be imposed at one cross-section and which is transmitted instantaneously along each fibre. If one cross-section X_1 = constant is fixed, or if the deformation has symmetry about some value of s, the shear $\Gamma(X_2,t)$ is zero and (12) becomes $\gamma = \theta$.

Using (12) in (10) gives

$$\ell(s,X_2,t) = 1 - X_2 \frac{\partial \Theta}{\partial s}\,, \tag{13}$$

since the condition $\ell(s,0,t) = 1$ follows from $X(s,0,t) = s$. Similarly when the extra shear Γ is written as

$$\Gamma(X_2,t) = \frac{\partial \psi}{\partial X_2}\,,$$

where $\psi = \psi(X_2,t)$ satisfies $\psi(0,t) = 0$, then the equations

$$\frac{\partial m}{\partial X_2} = -\frac{\partial^2 \psi}{\partial X_2 \partial t} - \frac{\partial \Theta}{\partial t} \qquad \text{and} \qquad m(s,0,t) = 0$$

give

$$m(s,X_2,t) = -\frac{\partial \psi}{\partial t} - X_2 \frac{\partial \Theta}{\partial t}\,. \tag{14}$$

In fibre-normal coordinates the compatibility equations (4_2) for the velocities become

$$\frac{\partial v_i}{\partial s} = \ell \frac{\partial F_{i1}}{\partial t} - m \frac{\partial F_{i1}}{\partial s} \qquad \text{and} \qquad \ell \frac{\partial v_i}{\partial X_2} + \gamma \frac{\partial v_i}{\partial s} = \ell \frac{\partial F_{i2}}{\partial t} - m \frac{\partial F_{i1}}{\partial s}\,.$$

After substitution from equations (2) and (3) these yield

$$\begin{aligned} &u \frac{\partial \theta}{\partial s} + \frac{\partial v}{\partial s} - \ell \frac{\partial \theta}{\partial t} + m \frac{\partial \theta}{\partial s} = 0, \\ &\frac{\partial u}{\partial s} - v \frac{\partial \theta}{\partial s} = 0, \end{aligned} \tag{15}$$

and the pair of equations

$$\begin{aligned} &u \frac{\partial \theta}{\partial X_2} + \frac{\partial v}{\partial X_2} = 0, \\ &\ell \left(\frac{\partial u}{\partial X_2} - v \frac{\partial \theta}{\partial X_2} \right) - v \frac{\partial(\gamma - \theta)}{\partial t} + m \frac{\partial(\gamma - \theta)}{\partial s} = 0. \end{aligned} \tag{16}$$

Using equation (11) we find from (16) that

$$\frac{\partial v}{\partial X_2} = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial X_2} = \frac{\partial \Gamma}{\partial t} = \frac{\partial^2 \psi}{\partial t \partial X_2} \, .$$

Consequently the velocity components u, v at each point are related to the longitudinal and transverse components U, V of the velocity of the central fibre $X_2 = 0$ by

$$v = V(s,t) \qquad \text{and} \qquad u(s,X_2,t) = U(s,t) + \frac{\partial \psi}{\partial t} \, . \tag{17}$$

When expressions (13), (14) and (17) are substituted into (15) they give

$$\frac{\partial U}{\partial s} - V \frac{\partial \Theta}{\partial s} = 0 \tag{18}$$

and

$$U \frac{\partial \Theta}{\partial s} + \frac{\partial V}{\partial s} - \frac{\partial \Theta}{\partial t} = 0. \tag{19}$$

These are just the compatibility conditions ensuring that the central fibre does not stretch, and that $\Theta(s,t)$ is its inclination to the x_1 axis.

4. The momentum equations

If $\mathbf{t} = (t_{ij})$ is the two-dimensional Cauchy stress ($i,j = 1,2$), the reaction stresses to the constraints $AB = 1$ and $A = 1$ are known to be (see Spencer [9] p. 53) respectively an arbitrary hydrostatic pressure and arbitrary extra tension along the fibre direction. Since $\mathbf{t}$ is symmetric, and since $F_{11} = \cos\theta$, $F_{21} = \sin\theta$ are the components of the unit vector along the fibres, the Cauchy stress may be written as

$$\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \left[\begin{pmatrix} 0 & G \\ G & 0 \end{pmatrix} - p \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

where p is a hydrostatic pressure and T the extra tension acting in the fibre direction. G is the shearing stress acting across the fibres X_2 = constant. In elastic materials G depends only on γ, whilst in viscoelastic and plastic materials it depends on the history of γ.

To analyse finite dynamic disturbances we find it convenient to use the Piola-Kirchhoff stress $\mathbf{T}$, given by

$$\mathbf{T} = (\det \mathbf{F})\, \mathbf{t}(\mathbf{F}^T)^{-1},$$

where $\mathbf{F} = (F_{ij})$ and $\mathbf{F}^T$ denotes its transpose. Then

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \left[G \begin{pmatrix} -\gamma & 1 \\ 1 & 0 \end{pmatrix} - p \begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix} + T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right]. \tag{20}$$

The two-dimensional momentum equations, in the absence of body forces, are given in Lagrangian coordinates by

$$\frac{\partial T_{i1}}{\partial X_1} + \frac{\partial T_{i2}}{\partial X_2} = \rho \frac{\partial v_i}{\partial t} \qquad \text{for} \qquad i = 1, 2,$$

and so in fibre-normal coordinates they take the form

$$\frac{\partial T_{i1}}{\partial s} + \frac{\partial T_{i2}}{\partial X_2} + \gamma \frac{\partial T_{i2}}{\partial s} = \rho \ell \frac{\partial v_i}{\partial t} - \rho m \frac{\partial v_i}{\partial s}, \tag{21}$$

where ρ is the uniform density in the reference configuration. When T_{ij} and v_i are expressed as in (20) and (3), and equations (7) and (11) are utilised, equation (21) yields the momentum equations in the fibre and normal directions as

$$\frac{\partial}{\partial s}(T - p) - 2G \frac{\partial \Theta}{\partial s} + \ell \frac{\partial G}{\partial X_2} = \rho \ell \left(\frac{\partial u}{\partial t} - v \frac{\partial \Theta}{\partial t} \right) \tag{22}$$

and

$$T \frac{\partial \Theta}{\partial s} - \ell \frac{\partial p}{\partial X_2} + \frac{\partial G}{\partial s} = \rho \ell \left(u \frac{\partial \Theta}{\partial t} + \frac{\partial v}{\partial t} \right) - \rho m \left(u \frac{\partial \Theta}{\partial s} + \frac{\partial v}{\partial s} \right). \tag{23}$$

Since any motion is completely expressible in terms of the motion $U(s,t)$, $V(s,t)$, $\Theta(s,t)$ of the central fibre and the function $\psi(X_2,t)$ which determines the extra shear, equation (22) may be regarded as an equation governing the total tension $(T-p)$ along material curves X_2 = constant at each instant of a kinematically allowable deformation. Similarly, by use of (13), equation (23) may be rewritten as

$$\frac{\partial(\ell p)}{\partial X_2} = (T - p) \frac{\partial \Theta}{\partial s} + \frac{\partial G}{\partial s} - \rho \ell \left(u \frac{\partial \Theta}{\partial t} + \frac{\partial v}{\partial t} \right) + \rho m \left(u \frac{\partial \Theta}{\partial s} + \frac{\partial v}{\partial s} \right), \tag{23a}$$

which determines the pressure variation along each normal. Consequently, equations (22) and (23a) are merely characteristic equations for $T-p$ and for p in any admissible motion. The corresponding characteristic speeds are both infinite. Thus *every kinematically admissible plane strain disturbance in an ideal fibre-reinforced material is also mechanically admissible*, and is attainable when the surface tractions are suitably specified. This is an extension to dynamic deformations of a result in the static theory due to Pipkin and Rogers [4].

5. Surface layers

Let $P^{\pm}(s,t)$ and $Q^{\pm}(s,t)$ denote the tangential and normal tractions applied over the surfaces $X_2 = \pm 1$. The the representation (20) for the stress within the slab predicts that G and p should satisfy

$$G(s,\pm 1,t) = P^{\pm}(s,t), \qquad p(s,\pm 1,t) = -\,Q^{\pm}(s,t). \tag{24}$$

In an elastic material the shear stress G is given by some function $G = G(\gamma)$, whilst (12) implies that $\gamma - \Theta(s,t)$ is constant along each of the surfaces $X_2 = \pm 1$. In general this is not compatible with the applied shear-tractions $P^{\pm}(s,t)$. For example, the surfaces could not be simultaneously traction-free ($P^+ = 0 = P^-$) over any portion of the slab which is curved.

The resolution of this paradox, as in similar situations arising in static problems, is to include concentrated loads acting along the boundaries [4]. In reality, such loads are distributed through thin layers adjacent to the boundaries, where the ideal assumptions $A = 1, AB = 1$ are only approximately valid. The consequent high tensions $(T-p)$ along the fibres are balanced in equation (22) by large values of $\partial G/\partial X_2$. The severe gradients of shear allow G to adjust to the boundary condition (24_1) through a 'boundary layer' as has been demonstrated for linearised static deformations by Everstine and Pipkin [5] and by Spencer [6]. Analysis of dynamic deformations using the present formulation [7] shows that inertia effects are unimportant within the boundary layers.

Let $L^+(s,t)$ and $L^-(s,t)$, defined by

$$L^+ \equiv \int_{1-}^{1} (T-p)\mathrm{d}X_2, \qquad L^- \equiv \int_{-1}^{-1+} (T-p)\mathrm{d}X_2,$$

be the concentrated loads carried by the layers immediately adjacent to the surfaces $X_2 = \pm 1$ respectively. Similarly, let $G^+(s,t)$, $G^-(s,t)$, $p^+(s,t)$ and $p^-(s,t)$ denote the shear stress and pressure immediately adjacent to each of these layers. Integration of equation (22) through the boundary layers then gives

$$\frac{\partial L^+}{\partial s} = \frac{\partial}{\partial s}\int_{1-}^{1} (T-p)dX_2 = -\ell(s,1,t)\int_{1-}^{1}\frac{\partial G}{\partial X_2}\,dX_2 = \ell(s,1,t)\,[G^+(s,t) - P^+(s,t)]$$
$$= (1-\Theta_s)\,[G^+(s,t) - P^+(s,t)],$$

where $\Theta_s(s,t) \equiv \partial\Theta/\partial s$, and similarly

$$\frac{\partial L^-}{\partial s} = -\,\ell(s,-1,t)\int_{-1}^{-1+}\frac{\partial G}{\partial X_2}\,dX_2 = -\,(1+\Theta_s)\,[G^-(s,t) - P^-(s,t)].$$

Thus, the loads $L^{\pm}$ vary according to

$$\frac{\partial L^{\pm}}{\partial s} = \pm(1 \mp \Theta_s)\,[G^{\pm}(s,t) - P^{\pm}(s,t)]. \tag{25}$$

This may be written more compactly as

$$\frac{\partial L^{\pm}}{\partial X_1} = \pm (G^{\pm} - P^{\pm}) \tag{25a}$$

and shows that, as in the static theory, the variation of the load along the boundary layer is exactly balanced by the resultant shearing traction.

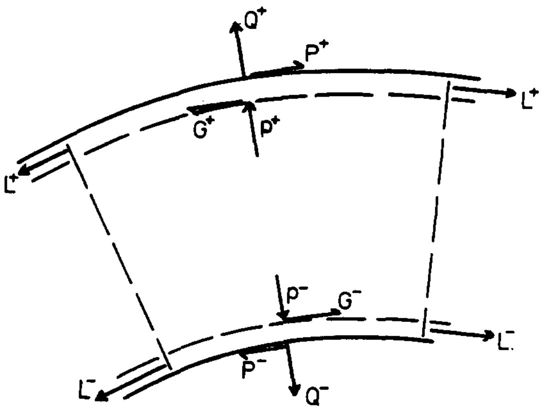

Figure 3. Tractions acting on elements of the boundary layers at $X_2 = 1$ and $X_2 = -1$, and the resultant loads L^+ and L^- within these layers.

As may be seen from Fig. 3, any curvature of the boundary must be accompanied by a jump in normal stress. This jump is determined, by integration of (23a) through either layer, from

$$\Omega(s,1,t)\,[p^+(s,t) + Q^+(s,t)] = -\int_{1-}^{1} \frac{\partial}{\partial X_2}(\Omega p)dX_2 = -\Theta_s \int_{1-}^{1}(T-p)dX_2 = -\Theta_s L^+$$

and a similar equation near $X_2 = -1$. Consequently, using (9) and (13) which give

$$\Omega(s,\pm 1,t) = \frac{\partial X}{\partial s} = 1 \mp \Theta_s,$$

we relate the interior pressure p to $Q^{\pm}(s,t)$ by

$$p = p^{\pm}(s,t) = -Q^{\pm}(s,t) \mp \frac{\Theta_s}{1 \mp \Theta_s} L^{\pm}(s,t). \tag{26}$$

Like (25) this equation may be written more concisely as

$$p^{\pm} = -Q^{\pm} \mp L^{\pm}\frac{\partial \theta}{\partial X_1} \qquad \text{on} \qquad X_2 = \pm 1. \tag{26a}$$

Equations (25a) and (26a) are identical with those for static stress concentration layers. The reason is that, in the ideal theory, the layers are negligibly thin and so inertia effects do not arise [7].

6. Flexural waves

The motion of the slab is entirely determined by the orientation $\Theta(s,t)$ of the central fibre, by its associated velocities $U(s,t)$ and $V(s,t)$, and by the extra shear $\psi(X_2,t)$. To obtain governing equations for these we first integrate the momentum equations (22) and (23a) over the interior region from $X_2 = -1+$ to $X_2 = 1-$. Substitution for ℓ, m, u, v and θ from (11), (13), (14) and (17) gives

$$\frac{\partial}{\partial s}\int_{-1+}^{1-}(T-p)dX_2 - \Theta_s\int_{-1+}^{1-}GdX_2 + [\ell G]_{-1+}^{1-} = 2\rho(U_t - V\Theta_t) + \rho\int_{-1+}^{1-}(1 - X_2\Theta_s)\frac{\partial^2\psi}{\partial t^2}dX_2$$

and

$$\ell(s,1,t)p^+ - \ell(s,-1,t)p^- = \Theta_s\int_{-1+}^{1-}(T-p)dX_2 + \frac{\partial}{\partial s}\int_{-1+}^{1-}GdX_2 - 2\rho(U\Theta_t + V_t) - 2\rho\Theta_t\int_{-1+}^{1-}\frac{\partial\psi}{\partial t}dX_2 - \rho\Theta_s\int_{-1+}^{1-}\left(\frac{\partial\psi}{\partial t}\right)^2 dX_2,$$

where equation (19) has been used, and $\Theta_t \equiv \partial\Theta/\partial t$. These equations may be written in terms of the resultant longitudinal load

$$R(s,t) \equiv \int_{-1+}^{1-}(T-p)dX_2 + L^-(s,t) + L^+(s,t), \tag{27}$$

and the resultant shearing load

$$S(s,t) = \int_{-1}^{1}GdX_2 = \int_{-1+}^{1-}GdX_2, \tag{28}$$

by use of the results (25) and (26). When subscripts s and t denote partial derivatives, and dots denote time derivatives, we obtain

$$2\rho(U_t - V\Theta_t) = R_s - S\Theta_s - \dot{F}_0 + \Theta_s\dot{F}_1 + P^+(1 - \Theta_s) - P^-(1 + \Theta_s), \tag{29}$$

and

$$2\rho(U\Theta_t + V_t) = R\Theta_s + S_s - 2\Theta_t F_0 - \Theta_s F_2 + Q^+(1 - \Theta_s) - Q^-(1 + \Theta_s). \tag{30}$$

Here

$$F_0(t) \equiv \rho \int_{-1}^{1} \psi_t(X_2,t)dX_2, \qquad F_1(t) \equiv \rho \int_{-1}^{1} X_2\psi_t(X_2,t)dX_2$$

and

$$F_2(t) \equiv \rho \int_{-1}^{1} [\psi_t(X_2,t)]^2 \, dX_2 \tag{31}$$

are functions of the momenta associated with the extra shearing displacement $\psi(X_2,t)$, and all disappear when ψ is independent of time.

Equations (18), (19), (29) and (30), together with (28) which relates S to $\Theta(s,t)$ through a constitutive law expressing G in terms of the history of $\gamma = \Theta(s,t) + \partial\psi/\partial X_2$, form a system of equations governing $U(s,t)$, $V(s,t)$, $\Theta(s,t)$ and $R(s,t)$. (For static deformations, the system is equivalent to those treated in [10], [11]). The extra shear $\psi(X_2,t)$ propagates instantaneously along each fibre, and since it is determined by conditions on any one cross-section of the slab it may be regarded as a forcing function for the system of equations. Likewise, changes in R propagate instantaneously along the slab, since equation (29) is the only equation to contain a derivative of $R(s,t)$.

The complete classification of the system depends on the constitutive assumptions relating G to γ. In this paper only *elastic materials* will be considered, so that $G = G(\gamma)$. The shearing load is then expressible as

$$S = \int_{-1}^{1} G(\Theta + \partial\psi/\partial X_2)dX_2 \equiv S(\Theta,t), \tag{32}$$

where the explicit dependence on t arises from the dependence on $\psi(X_2,t)$. (For deformations in which $\psi_t \equiv 0$, equation (32) reduces to $S = S(\Theta)$, whilst for deformations in which $\psi \equiv 0$ it reduces to $S = 2G(\Theta)$). Substitution of (32) into (30) gives

$$2(\rho U + F_0)\Theta_t + 2\rho V_t = (R + S_\Theta - F_2 - Q^+ - Q^-)\Theta_s + (Q^+ - Q^-)$$

where $S_\Theta \equiv \partial S/\partial\Theta$. Whenever

$$R + S_\Theta > Q^+ + Q^- + F_2 - \tfrac{1}{2}\rho^{-1}F_0^2, \tag{33}$$

this equation may be combined with (19) to give the two characteristic differential equations

$$(U + c_2)\frac{\partial\Theta}{\partial t} + \frac{\partial V}{\partial t} + c_1\left\{(U + c_2)\frac{\partial\Theta}{\partial s} + \frac{\partial V}{\partial s}\right\} = \frac{Q^+ - Q^-}{2\rho} \tag{34}$$

and

$$(U + c_1)\frac{\partial\Theta}{\partial t} + \frac{\partial V}{\partial t} + c_2\left\{(U + c_1)\frac{\partial\Theta}{\partial s} + \frac{\partial V}{\partial s}\right\} = \frac{Q^+ - Q^-}{2\rho}, \tag{35}$$

where the characteristic speeds c_1 and c_2 are given by

$$c_1, c_2 = \tfrac{1}{2}\rho^{-1}F_0(t) \pm \tfrac{1}{2}\rho^{-1}\{2\rho(R + S_\Theta - Q^+ - Q^- - F_2) + F_0^2\}^{\frac{1}{2}}.$$

In this case the motion is governed by a system of hyperbolic equations, so that the Cauchy initial value problem is well-posed for arbitrary choice of initial deformation $\Theta(s, 0)$ and velocity $V(s,0)$. However, when the inequality (33) is violated the problem is ill-posed, and instabilities may develop. Such instabilities become checked only when R again becomes sufficiently large to satisfy (33). For the case $\psi \equiv 0$, $Q^+ + Q^- \equiv 0$ condition (33) becomes the static buckling condition of Kao and Pipkin [8], and states that the compressive load per unit area of cross-section shall not exceed the shear modules $\frac{1}{2}S_\Theta = G'(\gamma)$. A mean positive pressure $(-\frac{1}{2}(Q^+ + Q^-) > 0)$ applied over the bounding surfaces serves as an additional stabilising factor.

7. Simple waves

When the inequality (33) is satisfied the flexural elastic disturbances are wavelike, and are governed by equations (18), (19), (29) and (30). These are non-linear equations. Moreover, unless $\psi_t(X_2,t) = 0$, the coefficients and the function $S(\Theta,t)$ depend explicitly on time. Some features of the solutions are exhibited by the exact solutions which may be found when the extra shear ψ is independent of time and the applied tractions P^+, P^-, Q^+ and Q^- all vanish. These solutions are *simple waves.*

When $\psi_t \equiv 0$ and $S = S(\Theta)$ the governing equations reduce to the system of quasi-linear partial differential equations

$$\begin{aligned} U_s - V\Theta_s &= 0,\\ U\Theta_s + V_s - \Theta_t &= 0,\\ R_s - S(\Theta)\Theta_s - 2\rho(U_t - V\Theta_t) &= 0,\\ R\Theta_s + S'(\Theta)\Theta_s - 2\rho(U\Theta_t + V_t) &= 0, \end{aligned} \tag{36}$$

which is homogeneous in derivatives with respect to s and t. Any such system possesses solutions in which all dependent variables are functions of a single combination $\eta(s,t)$ of s and t. The *wavelets*

$$\eta(s,t) = \text{constant}$$

travel with speed $c = -\eta_t/\eta_s$ which is a characteristic speed of the system (36), and which may itself depend on η. The structure of the wave is found by substituting the assumptions $U = U(\eta)$, $V = V(\eta)$, $\Theta = \Theta(\eta)$ and $R = R(\eta)$ into the system of equations (36). The resulting set of equations is compatible when the speed c is one of the characteristic speeds c_1, c_2, which in this case become

$$c = c(\eta) = \pm (2\rho)^{-\frac{1}{2}} [R + S'(\Theta)]^{\frac{1}{2}} = c_1, c_2, \tag{37}$$

and when U, V, Θ and R are related by

$$R'(\eta) = S(\Theta)\Theta'(\eta), \qquad U'(\eta) = V\Theta'(\eta)$$

and

$$V'(\eta) = -[U + c(\eta)]\Theta'(\eta). \tag{38}$$

The positive and negative signs in (37) apply to waves travelling in the directions of increasing and decreasing s respectively. (Additionally there are solutions (41) corresponding to $\eta_s = 0$).

For each choice of sign in (37) the equations (38) express U, V and R as functions of Θ, involving three arbitrary constants of integration. Correspondingly $c = c(\eta)$ is given by (37), and the wavelets are given, without loss of generality, in the parametric form

$$s - c(\eta)t = \eta. \tag{39}$$

In (39), the label η denotes the position $s = \eta(s,0)$ of each wavelet at $t = 0$. The simple waves then describe the propagation of flexural disturbances in which the central fibre has arbitrary initial shape $\Theta(\eta) \equiv \Theta(s,0)$, provided that $U(\eta)$, $V(\eta)$ and $R(\eta)$ are related to $\Theta(\eta)$ as in (38). In the disturbance, U, V, R and Θ are each constant along the wavelets shown on an s, t diagram in Fig. 4. Unless c is constant these wavelets will tend to disperse in regions were $c'(\eta) > 0$, but to coalesce in regions where $c'(\eta) < 0$. The curvature of the central fibre then varies as the wave progresses, and is given by

$$\frac{\partial\Theta}{\partial s} = \Theta'(\eta)\,\frac{\partial\eta}{\partial s} = \frac{\Theta'(\eta)}{1 + c'(\eta)t}\,. \tag{40}$$

The tendency for 'shock formation' in regions of negative $c'(\eta)$ is a familiar feature of non-linear waves [12]. Usually, shocks first form at the instant when the wavelets in the s,t diagram begin to intersect. This situation can never arise in the present context, since the fibre-normals cannot intersect within the slab. Consequently $\ell(s,X_2,t)$ must nowhere become negative in $|X_2| \leqslant 1$, and 'shocks' will be initiated within solutions of (36) whenever $|\partial\Theta/\partial s|$ reaches unity. The structure of 'shocks', and the corresponding rules for their propagation are derived in the next section.

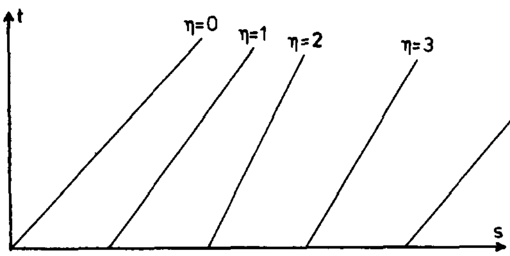

Figure 4. The propagation of wavelets η = constant within a simple wave, for the case $c > 0$.

In materials for which $c'(\eta) \equiv 0$ there is no tendency for shock formation, and equations (38) are readily integrated. These solutions have

$$S(\Theta) = \mu \sin(\Theta + \delta), \qquad R(\Theta) = k - \mu \cos(\Theta + \delta)$$

where each wavelet has the same propagation speed $c = \pm(k/2\rho)^{\frac{1}{2}}$. They describe simple waves which travel *without distortion* at speed c, with profile $\Theta = \Theta(\eta) = \Theta(s-ct)$ which is arbitrary provided that it satisfies the condition $|\Theta'(\eta)| < 1$. Notice that the speed c is independent of the constants μ and δ which characterise the constitutive law and the extra shear $\psi = \psi(X_2)$, but depends on k which is a constant of integration determined by the boundary conditions for R. Within these non-distorting waves the velocity of the central fibre has intrinsic components

$$U = V_0 \cos(\Theta - \epsilon) - c, \qquad V = -V_0 \sin(\Theta - \epsilon)$$

which correspond to the cartesian components

$$U \cos\Theta - V \sin\Theta = V_0 \cos\epsilon - c \cos\Theta, \qquad U \sin\Theta + V \cos\Theta = V_0 \sin\epsilon - c \sin\Theta.$$

Since solutions of non-linear equations cannot be superposed, simple waves cannot persist once some other disturbance begins to interact. This observation is particularly relevant for the waves (38), (39) since the system (36) possesses the characteristics t = constant. Unless U and R are correctly specified at a reference cross-section for all t, an interacting disturbance will travel infinitely fast through the simple wave. However, under suitable conditions a disturbance advancing into an undistorted region of the slab will be a simple wave. In other situations, the interaction region may be treated approximately using the techniques of modulated simple waves ([13], [14], [15]). In all cases, the disturbance emerging ahead of the flexural wave has the simple form

$$V = 0, \qquad \Theta = 0, \qquad U = U(t), \qquad R = R_0(t) + 2\rho s \dot{U}(t) \tag{41}$$

which corresponds to the infinite characteristic speed. It describes a longitudinal rigid body motion, with stress resultant $R(t)$ related to the acceleration $\dot{U}(t)$.

8. Fan regions

Discussion of the simple waves (38) shows that disturbances propagating according to (36) or to (18), (19), (29) and (30) may produce regions in which the normals intersect at one of the surfaces $X_2 = \pm 1$. Indeed, some initial conditions imply that such a *fan* forms immediately, either at one end of the beam, or near a point of concentrated loading. A *fan* has normals which meet at a *crease* on one surface of the beam, and will generally travel so as to separate two portions of the beam in which $|\Theta_s| < 1$. The equilibrium of static fans has been discussed by Pipkin and Rogers [4] and Everstine and Rogers [11], and applied to the solution of many problems (see [16]). The analysis for moving fans provides relations between disturbances on either side of the fan.

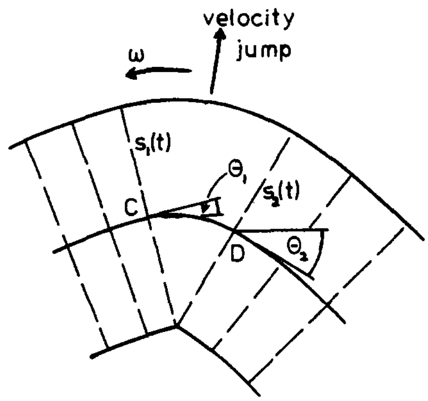

Figure 5. The configuration of fibres and normals in the vicinity of a fan $s_1(t) < s < s_2(t)$.

Consider the fan occupying a region covered by the normals through a moving portion $s_1(t) < s < s_2(t)$ of the central fibre, as in Fig. 5. Then within this region $\Theta_s = +1$ or -1, and the normals are centred at a crease on $X_2 = +1$ or $X_2 = -1$ respectively. Equations (18) and (19) imply that within the fan $\Theta(s,t) \mp s$ depends only on time and that at any instant all points of any fibre normal have equal velocities. Thus we may write

$$\Theta(s,t) \mp s = \Phi(t) = \Theta_1(t) \mp s_1(t) = \Theta_2(t) \mp s_2(t), \tag{42}$$

$$U(s,t) = q(t)\cos\Theta + r(t)\sin\Theta \pm \omega(t), \tag{43}$$

$$V(s,t) = -q(t)\sin\Theta + r(t)\cos\Theta, \tag{44}$$

where $q(t)$ and $r(t)$ are cartesian velocity components, and $\omega(t) \equiv \dot{\Phi}(t) = \Theta_t$ is the angular velocity associated with the rigid body motion of the central fibre. (Throughout Sections 8 and 9 the upper and lower signs correspond to $\Theta_s = +1$ and $\Theta_s = -1$ respectively, and any function $f_i(t)$ denotes the value $f(s_i(t),t)$ of a typical quantity $f(s,t)$ at the edges of the fan.) From (43) and (44) we deduce that the velocities U, V at either edge of the fan are related by

$$\begin{aligned} U_1 \mp \omega(t) &= [U_2 \mp \omega(t)]\cos(\Theta_2 - \Theta_1) - V_2 \sin(\Theta_2 - \Theta_1), \\ V_1 &= [U_2 \mp \omega(t)]\sin(\Theta_2 - \Theta_1) + V_2\cos(\Theta_2 - \Theta_1), \end{aligned} \tag{45}$$

where (42) gives

$$\Theta_2 - \Theta_1 = \pm[s_2(t) - s_1(t)].$$

Relations (45) confirm that the velocity of D relative to C is along the internal bisector of the angle of the fan and has the magnitude $2\,|\,\omega \sin \frac{1}{2}(\Theta_2 - \Theta_1)\,|$ associated with a rigid body motion having angular velocity $\omega(t)$, see Fig. 5.

Since the kinematics of the motion within the fan depends only on the three functions $\omega(t)$, $q(t)$ and $r(t)$, all effects are transmitted through the fan instantaneously. Equations (25) and (26) correctly give the pressure jump and boundary layer load at the outer surface $X_2 = \mp 1$, whilst equations (22) and (23) still determine T and p within the fan. However, (25) and (26) are not valid at the inner surface $X_2 = \pm 1$ where $\ell \to 0$ and the pressure and curvature become infinite. Consequently equation (23a) cannot be integrated from $X_2 = -1$ to $X_2 = 1$ within the fan, so that equation (30) must be discarded. No similar difficulty prevents integration of (22) with respect to X_2, so that equation (29) holds within the fan, and may be integrated with respect to s to give

$$R_2 - R_1 = \int_{\Theta_1}^{\Theta_2} G(\theta)d\theta \pm \rho[\dot{q}(t)\sin\Theta - \dot{r}(t)\cos\Theta]_{s_1}^{s_2} + \rho\dot{\omega}(t)(\Theta_2 - \Theta_1)$$
$$\pm \int_{s_1}^{s_2} P^{\mp}(s,t)ds,$$

in the case $\psi \equiv 0$. Use of (43), (44) shows that this equation becomes

$$R_2 - R_1 = \int_{\Theta_1}^{\Theta_2} G(\theta)d\theta + \rho\frac{d}{dt}[\omega(\Theta_2 - \Theta_1) \mp (V_2 - V_1)]$$
$$\mp \rho(U_2\dot{\Theta}_2(t) - U_1\dot{\Theta}_1(t)) \pm \int_{s_1}^{s_2} P^{\mp}(s,t)ds. \tag{46}$$

Thus, conditions on either side of the fan are related by equations (45) and (46). These are equivalent to equations (18), (19) and (29) together with the constraint equation $\Theta_s = \pm 1$.

Problems involving moving fans are free boundary problems. Equations (18), (19) and (29) hold everywhere, whilst (30) holds only when the strict inequality $|\Theta_s| < 1$ is satisfied. Otherwise it is replaced by the constraint equation $|\Theta_s| = 1$. The boundaries $s = s_1(t)$ and $s = s_2(t)$ of any region $|\Theta_s| = 1$ are determined as part of the solution. The quantities Θ, U, V and R are continuous, but their derivatives may be discontinuous.

9. Some special solutions

i) *Longitudinal disturbances*

Almost any disturbance imposed at $s = 0$ causes a longitudinal load $R(s,t)$ to be set up immediately in the portion $s > 0$ of the beam. The leading portion of flexural disturbance is some characteristic $ds/dt = c_1$, or possibly the front $s = s_2(t)$ of a fan. Ahead of this the disturbance is purely longitudinal with $\Theta \equiv 0$, $V \equiv 0$, so that (when $\psi \equiv 0$) the motion is a rigid body motion with speed $U = U(t)$ and load $R = \rho s\dot{U}(t) + R_0(t)$ as in (41). Consequently the longitudinal load in a disturbance neighbouring a free end $s = L$ is

$$R(s,t) = \rho(s - L)\dot{U}(t),$$

whilst in a disturbance approaching a bonded end $U(L,t) = 0$ it is

$$R = R_0(t).$$

ii) When a fan $s_1(t) \leqslant s \leqslant s_2(t)$ advances into a longitudinal region $s > s_2(t)$, the conditions ahead of the fan are

$$V_2 = 0, \qquad \Theta_2 = 0, \qquad U_2 = U(t), \qquad R_2 = R_2(t) = \rho(s_2 - L)\dot{U}(t) + R(L,t).$$

Consequently $\omega = \mp \dot{s}_2(t)$, $\Theta_1 = \pm [s_1(t) - s_2(t)]$ and the velocities behind the fan are

$$V_1 = -(U + \dot{s}_2)\sin\Theta_1, \qquad U_1 = -\dot{s}_2 + (U + \dot{s}_2)\cos\Theta_1.$$

The corresponding shear resultant is $S_1 = 2G(\Theta_1)$ whilst the longitudinal load is

$$R_1 = R_2(t) + 2\int_0^{\Theta_1} G(\theta)d\theta \mp \rho \frac{d}{dt}[\dot{s}_2\Theta_1 + V_1] \mp \rho U_1\dot{\Theta}_1.$$

These relations express conditions U_1, V_1, S_1 and R_1 behind the fan in terms of the angle Θ_1 of the fan and the speed $\dot{s}_2 = \dot{s}_1 \mp \dot{\Theta}_1(t)$ of the front of the fan, and the conditions ahead of the fan. Notice however that the formula for $R_1(t)$ involves first derivatives of some of these quantities, since the jump $R_2 - R_1$ in the resultant load is affected by the rate of change of momentum within the fan.

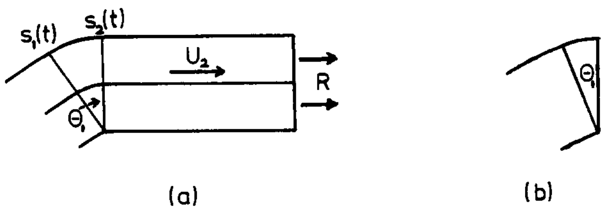

Figure 6. (a) A fan approaching a free end $X_1 = L$ through a region undergoing longitudinal motion with speed $U = U_2(t)$. (b) A fan at the end $X_1 = L$.

iii) In any problem with $\psi \equiv 0$, a free end $s = L$ has boundary condition $\Theta(L,t) = 0$. Whenever a flexural disturbance meets this end of the slab, a fan will form with $s_2(t) \equiv L$, $R_2 \equiv 0$, $\Theta_2 \equiv 0$, so that $\omega = 0$. The conditions at the rear $s = s_1(t)$ of the fan are then found from

$$U_1 = U_2\cos\Theta_1 + V_2\sin\Theta_1, \qquad V_1 = -U_2\sin\Theta_1 + V_2\cos\Theta_1$$

to be

$$S_1 = 2G(\Theta_1), \qquad R_1 = 2\int_0^{\Theta_1} G(\theta)d\theta \mp \rho(\dot{V}_1 - \dot{V}_2) \mp \rho U_1\dot{\Theta}_1,$$

where

$$V_1 - V_2 = V_1(1 - \cos\Theta_1) - U_1\sin\Theta_1.$$

These give conditions relating Θ_1, U_1, V_1, R_1 and S_1 at the boundary $s = s_1(t) = L \pm \Theta_1(t)$ of the region where the full equations (18), (19), (29) and (30) hold. Hence they determine the growth of fans at the ends of beams, and so control the consequent stress concentrations at the corners $X_2 = \pm 1$, $s = L$ of the slab. Such configurations require further investigation, since it is known that, even in isotropic materials, stress concentrations in corners are singular.

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